Mathematical transform for root helical patterns

TODO: REFERENCES

We propose to analyse the helical nature of root trajectories using the concepts of classical discrete Fourier transform theory. A root trajectory f is represented as a complex valued vectorial function,

$$\begin{aligned} f: \quad [0,T] &\longrightarrow \mathbb{C}^3 \\ t &\longrightarrow (f_x(t), f_y(t), f_z(t)), \end{aligned} \tag{1}$$

with an associated inner product $\langle f, g\rangle = 1/T \int_0^T (f_x, f_y, f_z/T^2) \cdot \overline{g} dt$, with functions f_x, f_y and f_z being square integrable L^2 functions. In order to extract the dominant wavelength and radius of helical shape of the root, we define a set of orthonormal spiral functions of various wavelength:

$$\psi_k(t) = \begin{pmatrix} e^{i2\pi kt} \\ e^{i(2\pi kt - \pi/2)} \\ 0 \end{pmatrix}, \text{ and } \psi_0(t) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{3}t \end{pmatrix}. \tag{2}$$

Because the family of function $(\psi_k)_{k\in\mathbb{N}}$ defines an orthonormal basis, i.e. $\langle \psi_j, \psi_k\rangle = \delta_{ij}$, an approximation of the root trajectory can be obtained by projection of the trajectory on the basis functions.

$$f(t) \approx \sum_{-N}^{+N} C_k \psi_k(t), \tag{3}$$

where C_k is determined as the projection of the trajectory on the k^{th} basis function

$$C_k = \langle f, \psi_k\rangle.$$

Coefficient C_k, for $k < 0$ indicate the magnitude of the anti-clockwise helical morphology while the coefficients C_k, for $k > 0$ indicate the magnitude of clockwise helical morphology. Because the image analysis process result in piecewise linear trajectory functions, N is chosen so that it corresponds to a bound of biologically relevant patterns. In this study, we did not consider frequency higher than 15 helical cycles per millimetre of root produced.

For the processing of the data, we choose $t = z$ so that $C_0 = 1/\sqrt{3}$. The frequency f of the spiral is the number of rotations of the helix when th depth of rooting is increased by a unit length, $f = 1/\lambda$ (Figure 1). The power spectrum is used to identify the direction of the rotation of the spiral:

$$P_k = C_k\overline{C_k}, k \in [-N, N], \tag{4}$$

and it is used in turn to express the radius spectrum according to the wavelength of the spiral

$$\lambda_k = 1/k, k \in [1, N], \tag{5}$$

$$r_k = \sqrt{P_k} + \sqrt{P_{-k}}, k \in [1, N].$$

The arclength s_k of the k^{th} helical waveform $\psi_0 + \psi_k$ if the length of the root after it has advanced by a unit length in the soil:

$$s_k = \sqrt{1 + (2\pi k r_k)^2}, k \in [-N, N], \tag{6}$$

and the curvature of the k^{th} helical waveform κ_k,

$$\kappa_k = \frac{r_k(2\pi k)^2}{1 + (2\pi k r_k)^2}, k \in [-N, N]. \tag{7}$$

It is possible to express the frequency $(\tilde{k})$ and wavelength $\tilde{\lambda}$ of the helix with regards to the length of roots being produce by growth:

$$\tilde{\lambda_k} = s_k \lambda_k \text{ and } \tilde{k} = k/s_k, k \in [-N, N]. \tag{8}$$

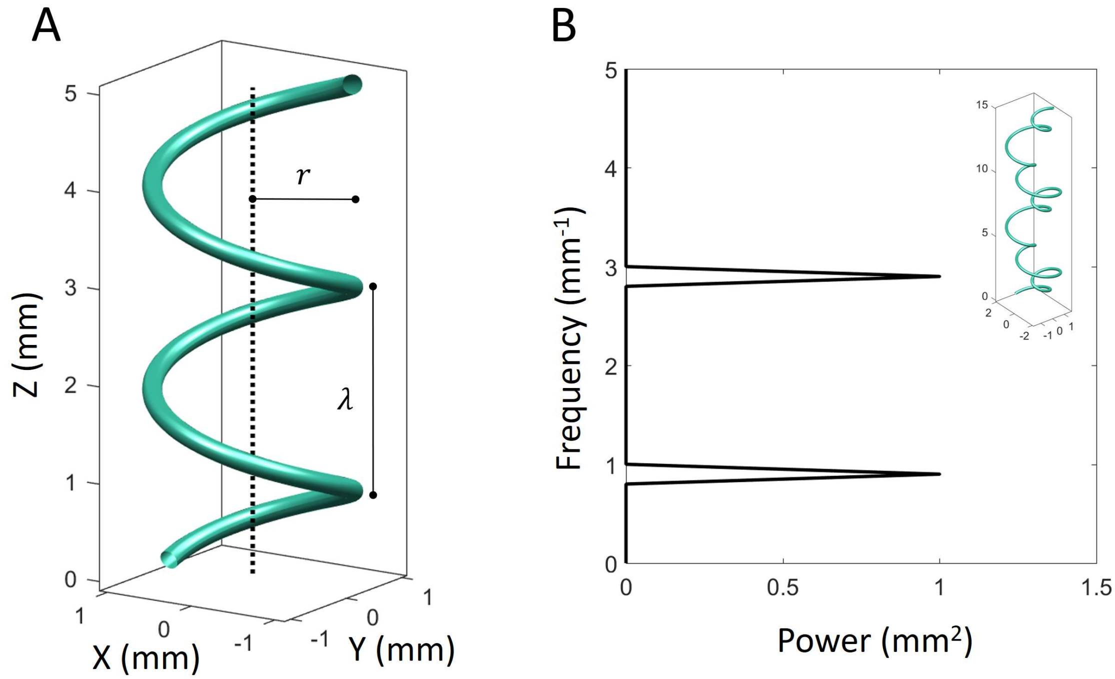

Figure 1: Helical transform. (A) An helix is a function f defined as $(x(t) = r\sin(2\pi kt)), y(t) = r\cos(2\pi kt), z(t) = k\lambda t)$, where r is the radius of the helix, k is its frequency, and λ its wavelength. (B) The helix transform decomposes a root trajectory into the contribution of various helical frequencies that compose it. In this example, a root trajectory is composed of two frequency $f(t) = \psi_1(t) + \psi_3(t)$